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SYMMETRIES IN OPTIMAL CONTROL*

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Abstract. It is argued that the existence of symmetries may simplify, as in classical mechanics, the solution of optimal control problems. A procedure for obtaining symmetries for the optimal Hamiltonian resulting from the Maximum Principle is given; this avoids the actual calculation of the optimal Hamiltonian. This procedure is based upon the notion of symmetry for the Hamiltonian system with inputs and outputs associated with an optimal control problem.

Key words. optimal control, Hamiltonian system, symmetry, reduction

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1. The Maximum Principle and Hamiltonian systems. Let us consider a smooth nonlinear control system

$$\dot{x} = f(x, u), \qquad x \in X, \quad u \in U \tag{1}$$

where f is a smooth mapping. (Smooth always means C^∞ or C^k with k "big enough.") For simplicity of exposition we will take $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$, although X and U may be arbitrary smooth manifolds. (We can even take the input space U to be *state dependent*. Then (x, u) are fiber respecting coordinates for a fiber bundle B over X, instead of coordinates for the product space $X \times U$, see [2], [6], [8].)

Let now $L: X \times U \to \mathbb{R}$ and $K: X \to \mathbb{R}$ be smooth functions. We consider the (unrestricted and smooth) Bolza problem of minimizing (with respect to $u(\cdot)$) the cost functional

$$J(x_0, u(\cdot)) = K(x(T)) + \int_0^T L(x(t), u(t))\, dt \tag{2}$$

under the constraints

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \in X. \tag{3}$$

This is called the (finite time) *optimal control problem.* Of course we have to worry about the class of functions $U(x_0)$ from $[0, T]$ to U (which may depend on the initial condition), over which the cost functional is minimized. Since we only want to deal with some *structural* properties of the above optimal control problem, we make the following simplifying assumptions (see also [4]):

1) $U(x_0)$ consists of measurable functions such that $\dot{x} = f(x, u)$ has a well-defined solution for all $t \in [0, T]$ and $x(0) = x_0$.

2) For each $x_0 \in X$ there exists a $u^*(\cdot) \in U(x_0)$ such that

$$J(x_0, u^*(\cdot)) = \min_{u(\cdot) \in U(x_0)} J(x_0, u(\cdot)), \tag{4}$$

($u^*(\cdot)$: $[0, T] \to U$ is called the *optimal control*).

In order to solve the optimal control problem, the Maximum Principle tells us to introduce the Hamiltonian function H: $X \times \mathbb{R}^n \times U \to \mathbb{R}$ given by

$$H(x, p, u) := p^T f(x, u) - L(x, u) \tag{5}$$

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with $p \in \mathbb{R}^n$ the *co-state*, and to consider the following set of differential equations

$$\text{(6a)} \qquad \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t), u(t)) = f_i(x(t), u(t)),$$

$$i = 1, \cdots, n,$$

$$\text{(6b)} \qquad \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), p(t), u(t)),$$

with the (mixed) boundary conditions

$$\text{(7)} \qquad \begin{aligned} &x(0) = x_0, \\ &p_i(T) = -\frac{\partial K}{\partial x_i}(x(T)), \qquad i = 1, \cdots, n, \end{aligned}$$

where $x(T)$ is the solution at time T of (6a) for $x(0) = x_0$. A necessary condition for a control function $u^* \in U(x_0)$ to be optimal, i.e., satisfying (4), is that for every $t \in [0, T]$

$$\text{(8)} \qquad H(x^*(t), p^*(t), u^*(t)) = \max_{u \in U} H(x^*(t), p^*(t), u)$$

where $(x^*(\cdot), p^*(\cdot))$ is the solution of (6) with $u(\cdot) = u^*(\cdot)$ and boundary conditions (7). So the Maximum Principle leads us to the following static optimization problem: Find for every $(x, p) \in X \times \mathbb{R}^n$ a $u^* \in U$ such that

$$\text{(9)} \qquad H(x, p, u^*) = \max_{u \in U} H(x, p, u).$$

Since we assumed U to be $\mathbb{R}^m$ (or a manifold), (9) implies the first order conditions

$$\text{(10)} \qquad \frac{\partial H}{\partial u_j}(x, p, u^*) = 0, \qquad j = 1, \cdots, m.$$

Hence the Maximum Principle leads in a natural way to the *system*

$$\text{(11)} \qquad \begin{aligned} &\dot{x}_i = \frac{\partial H}{\partial p_i}(x, p, u), \\ & && i = 1, \cdots, n, \\ &\dot{p}_i = -\frac{\partial H}{\partial x_i}(x, p, u), \\ &y_j = \frac{\partial H}{\partial u_j}(x, p, u), && j = 1, \cdots, m, \end{aligned}$$

and a necessary condition for $u^*(\cdot)$ to be optimal is that the *outputs* y_j of this system, resulting from $u^*(\cdot)$ and boundary conditions (7), are constant zero.

Now equations (11) form a *Hamiltonian system* as introduced in [2] and developed in [5], [6], [7], [8]. In fact the state space of this Hamiltonian system is $X \times \mathbb{R}^n = \mathbb{R}^{2n}$, with the natural symplectic form $\sum_{i=1}^n dp_i \wedge dx_i$, the input space is $U = \mathbb{R}^m$, and the output space is $Y = \mathbb{R}^m$, with coordinates $(y_1, \cdots, y_m)$. The product space $U \times Y$ has the natural symplectic form $\sum_{j=1}^m dy_j \wedge du_j$. From a geometric point of view equations (11) describe a $(2n+m)$-dimensional submanifold L of $T(X \times \mathbb{R}^n) \times (U \times Y)$ (the coordinates (x, p, u) parametrize the possible state space evolutions $(\dot{x}_i, \dot{p}_i)$ and outputs y_j). This submanifold has a special structure related to the given symplectic structures on $X \times \mathbb{R}^n$ and $U \times Y$.

Recall the definition of a *Lagrangian submanifold* [1], [8]. A submanifold L of a manifold N with symplectic form ω is Lagrangian if ω restricted to L is zero and $\dim L = \frac{1}{2}\dim N$.

DEFINITION 1 [5], [6], [7], [8]. A Hamiltonian system with state space $\mathbb{R}^n \times \mathbb{R}^n$, input space $\mathbb{R}^m$ and output space $\mathbb{R}^m$ is given by a submanifold $L \subset T(\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^m \times \mathbb{R}^m)$ such that

(i) L can be parametrized by the state space variables (x, p) and the input variables u.

(ii) L is a Lagrangian submanifold of $T(\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^m \times \mathbb{R}^m)$ with its natural symplectic form $\sum_{i=1}^n (d\dot{p}_i \wedge dx_i + dp_i \wedge d\dot{x}_i) - \sum_{j=1}^m dy_j \wedge du_j$.

Since L is Lagrangian and satisfies condition (i) there exists a *generating function* $H(x, p, u)$ for L such that L is given by equations (11) [7], [8]. In the optimal control case this generating function $H(x, p, u)$ has the extra property of being affine in the p-variables. We call (11) the Hamiltonian system *associated* with the optimal control problem.

Remark. If X and U are arbitrary manifolds, we have to generalize the definition of the associated Hamiltonian system in the following way. Instead of $\mathbb{R}^n \times \mathbb{R}^n$ we take as state space T^*X, and the space $\mathbb{R}^m \times \mathbb{R}^m$ of inputs and outputs becomes T^*U, where both cotangent bundles are endowed with their natural symplectic forms.

Now we investigate the consequences of imposing the necessary conditions (10) on the associated Hamiltonian system (11). Since the symplectic form $\sum_{i=1}^n (d\dot{p}_i \wedge dx_i + dp_i \wedge d\dot{x}_i) - \sum_{j=1}^m dy_j \wedge du_j$ is zero restricted to the submanifold L associated with (11), the form $\sum_{i=1}^n (d\dot{p}_i \wedge dx_i + dp_i \wedge d\dot{x}_i)$ is zero restricted to the subset $L \cap \{y_j = \partial H/\partial u_j = 0, j = 1, \cdots, m\}$, and therefore also restricted to the projection V of $L \cap \{y_j = 0, j = 1, \cdots, m\}$ onto $T\mathbb{R}^{2n}$. Now if V is a nice $2n$-dimensional submanifold of $T\mathbb{R}^{2n}$ it follows that V is a *Lagrangian* submanifold of $T\mathbb{R}^{2n}$, $\sum_{i=1}^n (d\dot{p}_i \wedge dx_i + dp_i \wedge d\dot{x}_i)$. Moreover, if V can be parametrized by the state space variables (x, p), this implies that V is actually the graph of a Hamiltonian vectorfield on $\mathbb{R}^{2n}$ [1], [8]. The simplest case is where the matrix $(\partial^2 H/\partial u_i\, \partial u_j)$ has rank m in every solution (x, p, u^*) of (10) (the so-called *nonsingular* case). Then the equations $\partial H/\partial u_j(x, p, u^*) = 0, j = 1, \cdots, m$, have locally a unique solution $u^*(x, p)$, and V is locally given as

$$V = \left\{ \left(x, p, \frac{\partial H}{\partial p}(x, p, u^*(x, p)), -\frac{\partial H}{\partial x}(x, p, u^*(x, p)) \right) \middle| x \in \mathbb{R}^n, p \in \mathbb{R}^n \right\}. \tag{12}$$

Hence V is locally the graph of the Hamiltonian vector field of the (locally defined) *optimal Hamiltonian* $H^0(x, p) := H(x, p, u^*(x, p))$.

Remark. If $(\partial^2 H/\partial u_i\, \partial u_j)$ is singular but the rank of the *map* $\partial H/\partial u(\cdot, \cdot, \cdot)$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^m$ is equal to m, then the set $(\partial H/\partial u)^{-1}(0)$ is a $2n$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Under certain regularity conditions (see [8]) on the map $(x, p, u) \to (x, p, \partial H/\partial p(x, p, u), -\partial H/\partial x(x, p, u))$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ to $T(\mathbb{R}^n \times \mathbb{R}^n)$ it follows that V is an (immersed) Lagrangian submanifold of $T(\mathbb{R}^n \times \mathbb{R}^n)$. However, since in general V need not be parametrized by the state space variables (x, p), we generally only obtain a set of *implicit* Hamiltonian differential equations on $\mathbb{R}^n \times \mathbb{R}^n$ [8].

For clarity of exposition we will make the following additional assumptions which will hold throughout the next section:

1) The matrix $(\partial^2 H/\partial u_i\, \partial u_j)(x, p, u^*)$ is nonsingular in every solution (x, p, u^*) of (10). The solution u^* of (10) is unique and is a smooth mapping $u^*(x, p)$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$.

2) This solution $u^*(x, p)$ is *optimal*, resulting in the optimal Hamiltonian $H^0(x, p) := H(x, p, u^*(x, p))$.

2. Symmetries. Under the simplifying assumptions made before the optimal control problem reduces to the solution of a set of Hamiltonian equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H^0}{\partial p_i}(x, p), \qquad x(0) = x_0, \\ \dot{p}_i &= -\frac{\partial H^0}{\partial x_i}(x, p), \quad p_i(T) = -\frac{\partial K}{\partial x_i}(x(T)), \end{aligned} \qquad i = 1, \cdots, n, \tag{13}$$

with $H^0(x, p) = H(x, p, u^*(x, p))$, where $u^*(x, p)$ is the unique solution of

$$\frac{\partial H}{\partial u_j}(x, p, u^*(x, p)) = 0, \qquad j = 1, \cdots, m. \tag{14}$$

Now solving (13) and (14) is typically a formidable task, and it is worthwhile to look for circumstances which make the solution easier.

If the equations (14) are explicitly solved for $u^*(x, p)$ and if we therefore have an *explicit* expression for $H^0(x, p)$, it is a classical method (in mechanics) to look for *symmetries* of H^0 in order to simplify the solution of (13). (This point was also raised in [3].) Let us introduce some notation. If $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then we denote the corresponding Hamiltonian vectorfield

$$\dot{x}_i = \frac{\partial F}{\partial p_i}(x, p), \quad \dot{p}_i = -\frac{\partial F}{\partial x_i}(x, p), \quad i = 1, \cdots, n \tag{15}$$

by X_F. Moreover if G is another smooth function on $\mathbb{R}^n \times \mathbb{R}^n$, then the *Poisson bracket* $\{F, G\}$ of F and G is defined as

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} \right). \tag{16}$$

It is easy to see that $\{F, G\} = -\{G, F\}$ and that

$$\{F, G\} = X_F(G). \tag{17}$$

Now the most general definition of an (infinitesimal) symmetry for H^0 is of a Hamiltonian vectorfield X_F satisfying

$$X_F(H^0) = \{F, H^0\} = 0. \tag{18}$$

This implies that $0 = \{F, H^0\} = -\{H^0, F\} = -X_{H^0}(F)$, and hence that F is a *first integral* or *conserved quantity* for (13). Therefore if X_F is a symmetry of H^0, then F is a first integral for X_{H^0}. Conversely if F is a first integral for X_{H^0}, i.e., $X_{H^0}(F) = 0$, it follows that X_F is a symmetry for H^0. The existence of such a conserved quantity F for (13) may be used for *reducing* the $2n$-dimensional set of equations (13) to a $(2n-2)$-dimensional set. Indeed, suppose that dF nowhere vanishes (this can be relaxed). Then there exists a constant c such that the solution of (13) remains within the submanifold $F^{-1}(c)$. Moreover we may factor out $F^{-1}(c)$ by the integral curves of X_F to obtain a $(2n-2)$-dimensional manifold. It follows from $X_F(H^0) = 0$ that the equations (13) project to Hamiltonian equations on this reduced manifold. If there are more symmetries available, or a *group* of symmetries, this reduction procedure can be generalized [1]. In general the existence of symmetries for H^0 reduces the solution of (13) to the solution of a *lower-dimensional* set of Hamiltonian equations. (Notice however that our situation is somewhat more complicated than in mechanics, since we do not know the initial conditions of (13), but a mixed set of initial and terminal conditions, see also [4].)

In conclusion, if we have an explicit expression for H^0 the knowledge of symmetries simplifies the solution $x^*(\cdot), p^*(\cdot)$ of (13). Henceforth it also simplifies the construction of the optimal control in open loop form $u^*(t)=u^*(x^*(t), p^*(t))$, or in feedback form $u^*(x^*(t), t)=u^*(x^*(t), p^*(t))$ from the solution $u^*(x, p)$ of (14).

Remark. A symmetry X_F for H^0 may be also profitably used for solving the Hamilton-Jacobi-Bellman equation

$$\begin{aligned}\frac{\partial S}{\partial t}(x, t) &= -\max_u \left(-L(x, u)+\frac{\partial S}{\partial x}(x, t)f(x, u)\right)\\ &= -H^0\left(x, \frac{\partial S}{\partial x}(x, t)\right), \qquad S(x, T)=-K(x).\end{aligned} \tag{19}$$

(We have adopted the sign convention from mechanics—S is minus the Bellman value function.) Now (19) defines a flow on the set of Lagrangian submanifolds of $\mathbb{R}^n \times \mathbb{R}^n$. In fact for every t the Lagrangian submanifold is given as $\{(x, p=(\partial S/\partial x)(x, t))\}$. If X_F is a symmetry for H^0, it follows that the action of X_F commutes with this flow. Explicitly, if the integral flow $X_F(\tau)$ of X_F maps for a *small* and *fixed* τ the Lagrangian submanifold $\{(x, p=(\partial S/\partial x)(x, T))\}$ onto another Lagrangian submanifold $\{(x, p=(\partial R/\partial x)(x))\}$, then $X_F(\tau)$ maps *for every* $t\in[0, T]$ the Lagrangian submanifold $\{(x, p=(\partial S/\partial x)(x, t))\}$ onto Lagrangian submanifolds $\{(x, p=(\partial R/\partial x)(x, t))\}$ where $R(x, t)$ is the solution of

$$\frac{\partial R}{\partial t}(x, t) = -H^0\left(x, \frac{\partial R}{\partial x}(x, t)\right), \qquad R(x, T)=R(x). \tag{20}$$

Therefore instead of solving (19) we may also solve (20) for the maybe easier terminal condition $R(x, T)=R(x)$. In the linear quadratic case (i.e. $F(x, u)=Ax+Bu$, $L(x, u)=\frac{1}{2}x^TQx+\frac{1}{2}u^TRu$, and (19) becoming a Riccati equation) this was noted in [10].

Of course in many cases an explicit expression for $u^*(x, p)$ and $H^0(x, p)$ is hard to obtain. However in the author's thesis [8] it was indicated that even *without* explicitly calculating $H^0(x, p)$ we can a priori deduce symmetries for $H^0(x, p)$ by looking for symmetries of the associated Hamiltonian *system* (11). The same idea was used by Grizzle and Marcus [4] from a different point of view. Let $\dot{x}=f(x, u)$, with $x\in M$ (n-dimensional) and $u\in U$, be a control system. Suppose there exists a vectorfield $G(x)$ on M such that $[G(x), f(x, u)]=0$ for all $u\in U$ (G is called a symmetry of the control system (cf. [8], [4]). Furthermore suppose that $G(L(x, u))=0$ for all $u\in U$ and that $G(K(x))=0$. (G is called a symmetry of the optimal control problem, [4].)

Then if G is nowhere zero, M can be locally factored out by the integral curves of G to obtain an $(n-1)$-dimensional manifold N. It is then shown [4] that the optimal control problem (2) reduces to an optimal control problem defined on this lower dimensional manifold N, and that the optimal control in feedback form can be defined on N. Furthermore this can be generalized from single vectorfields G to a Lie algebra of symmetry vectorfields generated by a symmetry Lie group.

In the sequel it will be shown that this kind of symmetry for the optimal control problem considered in [4] corresponds to a special, although important, type of symmetry for the associated Hamiltonian system and the optimal Hamiltonian $H^0(x, p)$. Furthermore if the end cost function K is *not* invariant under G ($G(K(x))\neq 0$) it is noted in [4] that the above procedure cannot be followed without modifications, but recourse has to be taken to the same Hamiltonian approach as will be used in this paper. On the other hand the class of symmetries considered in [4] is *enlarged* in [4] by allowing for *feedback* transformations. A feedback $u=\alpha(x, v)$ transforms the optimal control problem (2) into

$$\min_v K(x(T))+\int_0^T L(x(t), \alpha(x(t), v(t)))\,dt$$

under the constraints $\dot{x}(t)=f(x(t), \alpha(x(t), v(t)))$, $x(0)=x_0$. Now a vectorfield G may be a symmetry of this *transformed* optimal control problem without being a symmetry of the original optimal control problem. Such a symmetry G also results in a symmetry of the optimal Hamiltonian but may *not* be obtainable by our approach (although in most cases it will, see the examples). This leads to the question of determining *what* class of symmetries for H^0 can be obtained by our approach and how this class is affected by feedback. This problem is addressed (but not fully solved) in the last part of the paper.

We will now show how we can deduce symmetries for (13) *without* explicitly constructing $H^0(x, p)$ by looking for symmetries of the associated Hamiltonian system (11). Recall the notion of a *prolongation* of a vectorfield or a function. Let S be a vectorfield on M with integral flow S_t (i.e., $(d/dt)S_t(x)=S(S_t(x))$). Then $(S_t)_*: TM\to TM$ is the integral flow of a vectorfield on TM which we denote by $\dot{S}$. Let further $F: M\to\mathbb{R}$, then $\dot{F}: TM\to\mathbb{R}$ is defined by $\dot{F}(v)=dF(v)$, $v\in TM$.

DEFINITION 2 [5], [7], [8]. Let (11) be a Hamiltonian system given by a Lagrangian submanifold $L\subset T(\mathbb{R}^n\times\mathbb{R}^n)\times(\mathbb{R}^m\times\mathbb{R}^m)$. An (infinitesimal) *symmetry* is a pair of vectorfields (S, S^e), S a Hamiltonian vectorfield on $\mathbb{R}^n\times\mathbb{R}^n$ and S^e a Hamiltonian vectorfield on $\mathbb{R}^m\times\mathbb{R}^m$, such that the vectorfield $(\dot{S}, S^e)$ on $T(\mathbb{R}^n\times\mathbb{R}^n)\times(\mathbb{R}^m\times\mathbb{R}^m)$ is *tangent* to L, i.e., $(\dot{S}, S^e)(z)\in T_zL$ for all $z\in L$.

A *conservation law* is a pair of functions (F, F^e), with $F:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $F^e:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$, such that the function $\dot{F}-F^e: T(\mathbb{R}^n\times\mathbb{R}^n)\times(\mathbb{R}^m\times\mathbb{R}^m)\to\mathbb{R}$ restricted to L is zero.

Remark. The above definitions are really extensions of the usual definitions of symmetry and conserved quantity for Hamiltonian differential equations, as can be seen as follows. If we forget about inputs and outputs, so if $L\subset T(\mathbb{R}^n\times\mathbb{R}^n)$ is just the graph of a Hamiltonian vectorfield X_H, then $\dot{S}$ being tangent to L means the following. In coordinates $\dot{S}$ is given as $S(x)(\partial/\partial x)+(\partial S/\partial x)(x)\dot{x}(\partial/\partial\dot{x})$ (we forget about indices). Consider now a point $z=(x, X_H(x))$ on L. Elements of T_zL are of the form $(\partial/\partial x)+(\partial X_H(x)/\partial x)(\partial/\partial\dot{x})$. Hence $\dot{S}$ is tangent to L in z if $S(x)(\partial/\partial x)+(\partial S/\partial x)(x)X_H(x)(\partial/\partial\dot{x})$ is a multiple of $(\partial/\partial x)+(\partial X_H(x)/\partial x)(\partial/\partial\dot{x})$. This only happens if $(\partial S/\partial x)(x)X_H(x)=(\partial X_H(x))/(\partial x)S(x)$, or equivalently if the Lie bracket $[S, X_H]$ equals zero. Furthermore, $\dot{F}-F^e=0$ restricted to L just means that $dF/dt=F^e(y, u)$, with $y_j=\partial H/\partial u_j$, where d/dt is differentiation along the system (13).

As is the case for Hamiltonian *vectorfields*, symmetries and conservation laws for Hamiltonian *systems* are in one-to-one correspondence [5], [8]. In fact if (S, S^e) is a symmetry, then there exists a conservation law (F, F^e) such that $S=X_F$, $S^e=X_{F^e}$, and conversely if (F, F^e) is a conservation law, then (X_F, X_{F^e}) is a symmetry.

We notice that (F, F^e) being a conservation law for (11) can be also succinctly expressed by the equality (see (17))

$$\{H(x, p, u), F(x, p)\}=F^e\left(\frac{\partial H}{\partial u}(x, p, u), u\right)\quad \forall x, p, u \tag{21}$$

where $\{\,,\}$ means Poisson bracket on $\mathbb{R}^n\times\mathbb{R}^n$.

We now show how symmetries (or conservation laws) for the Hamiltonian system (11) yield symmetries (or conserved quantities) for the optimal Hamiltonian.

THEOREM 3 [8]. *Let* $(S = X_F, S^e = X_{F^e})$ *be a symmetry for* (11). *Then* S *is a symmetry for* H^0 *if* $F^e(0, u) = 0$, *for all* $u \in U$.

Proof. Since (X_F, X_{F^e}) is a symmetry, (21) holds. Therefore

$$\{H(x, p, u^*(x, p)), F(x, p)\} = F^e\left(\frac{\partial H}{\partial u}(x, p, u^*(x, p)), u^*(x, p)\right) + \sum_{j=1}^{m} \frac{\partial H}{\partial u_j}(x, p, u^*(x, p))\{u_j^*(x, p), F(x, p)\}.$$

Since $(\partial H/\partial u_j)(x, p, u^*(x, p)) = 0$, we obtain $\{H^0(x, p), F(x, p)\} = F^e(0, u^*(x, p))$, and hence if $F^e(0, u) = 0$, for all u, $\{H^0(x, p), F(x, p)\} = -S(H^0) = 0$. □

In conclusion, one can obtain symmetries of H^0 by looking for pairs (F, F^e) satisfying (21) and $F^e(0, u) = 0$. Furthermore, these symmetries may also be useful in finding the solution $u^*(x, p)$ of (14):

THEOREM 4. *Let* (X_F, X_{F^e}) *be a symmetry for* (11) *with* $F^e(0, u) = 0$ $\forall u$. *Let* $u^*(x, p) = (u_1^*(x, p), \cdots, u_m^*(x, p))$ *be the solution of* (14). *Then for* $j = 1, \cdots, m$

$$\{F(x, p), u_j^*(x, p)\} = \frac{\partial F^e}{\partial y_j}(0, u^*(x, p)). \tag{22}$$

Proof. Differentiate the equalities $(\partial H/\partial u_j)(x, p, u^*(x, p)) = 0$, $j = 1, \cdots, m$, with respect to x_k and p_k, $k = 1, \cdots, n$:

$$\frac{\partial^2 H}{\partial x_k\, \partial u_j}(x, p, u^*) + \sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i\, \partial u_j}(x, p, u^*)\frac{\partial u_i^*}{\partial x_k}(x, p) = 0, \tag{23}$$

$$\frac{\partial^2 H}{\partial p_k\, \partial u_j}(x, p, u^*) + \sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i\, \partial u_j}(x, p, u^*)\frac{\partial u_i^*}{\partial p_k}(x, p) = 0. \tag{24}$$

Furthermore, differentiate

$$\{H(x, p, u), F(x, p)\} = \sum_{k=1}^{n}\left(\frac{\partial H}{\partial p_k}\frac{\partial F}{\partial x_k} - \frac{\partial H}{\partial x_k}\frac{\partial F}{\partial p_k}\right) = F^e\left(\frac{\partial H}{\partial u}, u\right)$$

with respect to u_j, $j = 1, \cdots, m$:

$$\sum_{k=1}^{m}\left(\frac{\partial^2 H}{\partial u_j\, \partial p_k}\frac{\partial F}{\partial x_k} - \frac{\partial^2 H}{\partial u_j\, \partial x_k}\frac{\partial F}{\partial p_k}\right) = \frac{\partial}{\partial u_j}\left(F^e\left(\frac{\partial H}{\partial u}, u\right)\right). \tag{25}$$

Evaluate (25) in the points $(x, p, u^*(x, p))$, and substitute (23) and (24) into (25):

$$-\sum_{k=1}^{m}\sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i\, \partial u_j}\frac{\partial u_i^*}{\partial p_k}\frac{\partial F}{\partial x_k} + \sum_{k=1}^{n}\sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i\, \partial u_j}\frac{\partial u_i^*}{\partial x_k}\frac{\partial F}{\partial p_k} = \frac{\partial}{\partial u_j}\left(F^e\left(\frac{\partial H}{\partial u}, u\right)\right) \tag{26}$$

with everything evaluated in $(x, p, u^*(x, p))$. The left-hand side of (26) is equal to

$$\sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i\, \partial u_j}(x, p, u^*)\{F, u_i^*\}(x, p), \tag{27}$$

while the right-hand side of (26) equals

$$\begin{aligned} &\frac{\partial}{\partial u_j}\left(F^e\left(\frac{\partial H}{\partial u}, u\right)\right)\Bigg|_{u=u^*(x,p)} \\ &= \sum_{i=1}^{n} \frac{\partial F^e}{\partial y_i}\left(\frac{\partial H}{\partial u}, u\right)\Bigg|_{u=u^*} \frac{\partial^2 H}{\partial u_j\, \partial u_i}(x, p, u^*) + \frac{\partial F^e}{\partial u_j}\left(\frac{\partial H}{\partial u}, u\right)\Bigg|_{u=u^*} \\ &= \sum_{i=1}^{n} \frac{\partial^2 H}{\partial u_j\, \partial u_i}(x, p, u^*)\frac{\partial F^e}{\partial y_i}(0, u^*) \end{aligned} \tag{28}$$

since $F^e(0, u) = 0$ implies $\partial F^e/\partial u_j(0, u) = 0$, $j = 1, \cdots, m$.

Since $(\partial^2 H/\partial u_i\,\partial u_j)(x, p, u^*)$ is nonsingular, we obtain (22). □

Remark. $(\partial F^e/\partial y_j)(0, u^*)$ may also be written as $\{F^e(y, u), u_j\}_{\mathbb{R}^{2m}}(0, u^*)$, with $\{\,,\}_{\mathbb{R}^{2m}}$ the Poisson bracket on $\mathbb{R}^{2m}$ (given by $(\{G(y, u), K(y, u)\}_{\mathbb{R}^{2m}} = \sum_{j=1}^m ((\partial G/\partial y_j)(\partial K/\partial u_j) - (\partial G/\partial u_j)(\partial K/\partial y_j))))$. Hence (22) may be rewritten as $\{F, u_j^*\}_{\mathbb{R}^{2n}} = \{F^e, u_j\}_{\mathbb{R}^{2m}}(0, u^*)$.

Therefore, if (F, F^e) is a conservation law with $F^e(0, u) = 0$ the solution of (14) has to belong to the mappings $u^*(x, p)$ whose components satisfy the partial differential equations

$$\{F, u_j^*\} = G_j(u^*), \qquad j = 1, \cdots, m \tag{29}$$

with $G_j(u^*) = (\partial F^e/\partial y_j)(0, u^*)$.

An important special case of Theorem 3 are the conservation laws (F, F^e) with F^e *identically* zero. The fact that such conservation laws may exist is due to the possible nonminimality of the Hamiltonian system. Indeed, if $F^e = 0$, we obtain

$$\{H(x, p, u), F\} = 0 \quad \forall x, p, u. \tag{30}$$

Hence all integral curves of the Hamiltonian system (11) starting from a fixed initial condition u_0 remain within a submanifold $F^{-1}(c)$, with c a constant, and therefore the system is not "controllable." Moreover, it follows from (30) that

$$X_F(y_j) = \left\{F, \frac{\partial H}{\partial u_j}(x, p, u)\right\} = \frac{\partial}{\partial u_j}\{F, H(x, p, u)\} = 0 \tag{31}$$

and that

$$[X_{H(x,p,u)}, X_F] = 0 \quad \forall u. \tag{32}$$

(Recall the identity $[X_F, X_G] = X_{\{F,G\}}$ for arbitrary functions F, G on $\mathbb{R}^{2n}$, [1].) Hence the system is not "observable" and we may factor out the state space by the integral curves of X_F. (It follows from (32) that the vectorfields $X_{H(x,p,u)}$ leave these integral curves invariant.) For a more detailed treatment of these issues we refer to [6], [8]. It follows from Theorem 4 that if $F^e = 0$ then the optimal $u^*(x, p)$ satisfies

$$\{F, u_j^*\} = 0, \qquad j = 1, \cdots, m. \tag{33}$$

Therefore if we reduce the $2n$-dimensional state space to a $(2n-2)$-dimensional space as sketched above, the optimal $u(x, p)$ also projects to a mapping on this reduced space. The symmetries considered in [4] form a subclass of this special type. Indeed, let the vectorfield G satisfy $[G(x), f(x, u)] = 0$, for all $u \in U$, and $G(L(x, u)) = 0$. Then

$$\begin{aligned}\{H(x, p, u), p^T G(x)\} &= \{p^T f(x, u) - L(x, u), p^T G(x)\}\\ &= p^T \frac{\partial G}{\partial x}(x) f(x, u) - p^T \frac{\partial F}{\partial x}(x, u) G(x) - p^T \frac{\partial L}{\partial x}(x, u) G(x)\\ &= p^T [f(x, u), G(x)] - p^T G(L(x, u)) = 0.\end{aligned}$$

Hence $(p^T G(x), 0)$ is a conservation law for the associated Hamiltonian system, and $p^T G(x)$ is a conserved quantity for the optimal Hamiltonian H^0. In the physics literature a symmetry with conserved quantity of the form $p^T G(x)$ is called a *geometrical* symmetry (because the symmetry is induced by a vectorfield on the x-space), in contrast to a symmetry with a general conserved quantity $F(x, p)$, which is called a *dynamical* symmetry.

As in the case of Hamiltonian vector fields [1], the treatment of a single symmetry may be extended to *groups* of symmetries. In our context the basic observation in order to do so is the following.

THEOREM 5. *Let* (F_i, F_i^e), $i=1,2$, *be two conservation laws for* (11), *with* $F_i^e(0,u)=0$, *for all* u. *Then* $(\{F_1,F_2\}_{\mathbb{R}^{2n}}, \{F_1^e,F_2^e\}_{\mathbb{R}^{2m}})$ *is again a conservation law with* $\{F_1^e,F_2^e\}_{\mathbb{R}^{2m}}(0,u)=0$, *for all* u. (*As before* $\{\,,\}_{\mathbb{R}^{2n}}$ *and* $\{\,,\}_{\mathbb{R}^{2m}}$ *denote Poisson brackets on* $\mathbb{R}^{2n}$, *resp.* $\mathbb{R}^{2m}$.)

Proof. This can be proved by geometric considerations [8], but also by the following explicit calculation. We have $\{H(x,p,u), F_i(x,p)\}=F_i^e((\partial H/\partial u),u)$, $i=1,2$. Hence, by Jacobi's identity for the Poisson bracket,

$$\begin{aligned}\{H(x,p,u),\{F_1,F_2\}\}&=\{\{H(x,p,u),F_1\},F_2\}-\{\{H(x,p,u),F_2\},F_1\}\\&=\left\{F_1^e\left(\frac{\partial H}{\partial u},u\right),F_2\right\}-\left\{F_2^e\left(\frac{\partial H}{\partial u},u\right),F_1\right\}.\end{aligned}$$

Now

$$\left\{F_1^e\left(\frac{\partial H}{\partial u},u\right),F_2\right\}=\sum_{j=1}^m\frac{\partial F_1^e}{\partial y_j}\left(\frac{\partial H}{\partial u},u\right)\left\{\frac{\partial H}{\partial u_j},F_2\right\},$$

and

$$\begin{aligned}\left\{\frac{\partial H}{\partial u_j},F_2\right\}&=\frac{\partial}{\partial u_j}\{H,F_2\}=\frac{\partial}{\partial u_j}\left(F_2^e\left(\frac{\partial H}{\partial u},u\right)\right)\\&=\frac{\partial F_2^e}{\partial u_j}+\sum_{k=1}^m\frac{\partial F_2^e}{\partial y_k}\frac{\partial^2 H}{\partial u_j\,\partial u_k}.\end{aligned}$$

Hence

$$\begin{aligned}\{H(x,p,u),\{F_1,F_2\}_{\mathbb{R}^{2n}}\}&=\sum_{j=1}^m\frac{\partial F_1^e}{\partial y_j}\frac{\partial F_2^e}{\partial u_j}-\frac{\partial F_1^e}{\partial u_j}\frac{\partial F_2^e}{\partial y_j}\\&\quad+\sum_{j=1}^m\sum_{k=1}^m\frac{\partial F_1^e}{\partial y_j}\frac{\partial F_2^e}{\partial y_k}\frac{\partial^2 H}{\partial u_j\,\partial u_k}-\sum_{j=1}^m\sum_{k=1}^m\frac{\partial F_2^e}{\partial y_j}\frac{\partial F_1^e}{\partial y_k}\frac{\partial^2 H}{\partial u_j\,\partial u_k}\\&=\{F_1^e,F_2^e\}_{\mathbb{R}^{2m}}.\end{aligned}$$

Furthermore, $F_i^e(0,u)=0$ implies $(\partial F_i^e(0,u))/\partial u_j=0$, $j=1,\cdots,m$, and hence

$$\{F_1^e,F_2^e\}(0,u)=\sum_j\left(\frac{\partial F_1^e(0,u)}{\partial y_j}\frac{\partial F_2^e(0,u)}{\partial u_j}-\frac{\partial F_1^e(0,u)}{\partial u_j}\frac{\partial F_2^e(0,u)}{\partial y_j}\right)=0. \qquad \square$$

Therefore the mapping $F\mapsto F^e$, given by $\{H(x,p,u),F\}=F^e$, is an algebra morphism from functions on $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$ (with respect to the respective Poisson brackets). It also follows from Theorems 4 and 5 that in the case of two conservation laws (F_i, F_i^e) the optimal $u^*(x,p)$ also has to satisfy

$$\{\{F_1,F_2\}_{\mathbb{R}^{2n}},u_i^*\}_{\mathbb{R}^{2n}}=\{\{F_1^e,F_2^e\}_{\mathbb{R}^{2m}},u_i\}_{\mathbb{R}^{2m}}(0,u^*). \tag{34}$$

We will now give some illustrative examples of the theory developed above.

Example 1. First we treat the example dealt with in Grizzle and Marcus [4] in our framework. Consider a particle of unit mass in a planar inverse-square-law gravitational field, which has thrusters in the "x-y" directions. The equations of motion in rectangular coordinates are given as

$$(\dot q_1,\dot q_2,\dot v_1,\dot v_2)=(v_1,v_2,-q_1(q_1^2+q_2^2)^{-3/2}+u_1,-q_2(q_1^2+q_2^2)^{-3/2}+u_2)=f(x,u)$$

and are defined on $M=(\mathbb{R}^2\backslash\{0\})\times\mathbb{R}^2$ and $U=\mathbb{R}^2$.

Let us take $L(x, u)=\frac{1}{2}(u_1^2+u_2^2)$. An evident candidate for a symmetry vectorfield on $\mathbb{R}^2\backslash\{0\}$ is $q_1(\partial/\partial q_2)-q_2(\partial/\partial q_1)$ (infinitesimal rotation). This vectorfield is prolonged to the vectorfield $G=q_1(\partial/\partial q_2)-q_2(\partial/\partial q_1)+v_1(\partial/\partial v_2)-v_2(\partial/\partial v_1)$ on M. Denote $x_1=q_1$, $x_2=q_2$, $x_3=v_1$, $x_4=v_2$; then the corresponding Hamiltonian function is $p^TG(x)=-p_1x_2+p_2x_1-p_3x_4+p_4x_3$. Calculation yields

$$\begin{aligned}\{H(x,p,u),p^TG(x)\}&=\{p^Tf(x,u)-L(x,u),p^TG(x)\}\\&=\{p_1x_3+p_2x_4+p_3(-x_1(x_1^2+x_2^2)^{-3/2}+u_1)\\&\qquad+p_4(-x_2(x_1^2+x_2^2)^{-3/2}+u_2)\\&\qquad\qquad-\tfrac{1}{2}u_1^2-\tfrac{1}{2}u_2^2,-p_1x_2+p_2x_1-p_3x_4+p_4x_3\}\\&=\text{(since the gravitational field is rotation invariant)}\\&\quad\{p_3u_1+p_4u_2,-p_1x_2+p_2x_1-p_3x_4+p_4x_3\}\\&=u_1p_4-u_2p_3.\end{aligned}$$

Furthermore,

$$y_1=\frac{\partial H}{\partial u_1}=p_3-u_1,\qquad y_2=\frac{\partial H}{\partial u_2}=p_4-u_2.$$

Hence,

$$\{H(x,p,u),p^TG(x)\}=u_1(y_2+u_2)-u_2(y_1+u_1)=u_1y_2-u_2y_1. \tag{35}$$

Therefore $(F,F^e)=(-p_1x_2+p_2x_1-p_3x_4+p_4x_3,\ u_1y_2-u_2y_1)$ is a conservation law for the associated Hamiltonian system satisfying $F^e(0,u)=0$, for all u. Hence by Theorems 3 and 4

$$\begin{aligned}&\{H(x,p,u^*(x,p)),-p_1x_2+p_2x_1-p_3x_4+p_4x_3\}=0,\\&\{-p_1x_2+p_2x_1-p_3x_4+p_4x_3,u_1^*(x,p)\}=\frac{\partial F^e}{\partial y_1}(0,u^*)=-u_2^*(x,p),\\&\{-p_1x_2+p_2x_1-p_3x_4+p_4x_3,u_2^*(x,p)\}=\frac{\partial F^e}{\partial y_2}(0,u^*)=u_1^*(x,p).\end{aligned}\tag{36}$$

On the other hand, in the Grizzle-Marcus approach one first applies *feedback*

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=(q_1^2+q_1^2)^{-1/2}\begin{pmatrix}q_1&q_2\\-q_2&q_1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=\alpha(x,w)\tag{37}$$

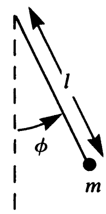
(defined on $\mathbb{R}^2\backslash\{0\}$), transforming $f(x,u)$ into $\tilde f(x,w)$, with $w=(w_1,w_2)$ the new inputs. The modified Hamiltonian $\tilde H(x,p,w)=p^T\tilde f(x,w)-\tilde L(x,w)$, with $\tilde L(x,w)=L(x,\alpha(x,w))=\frac{1}{2}w_1^2+\frac{1}{2}w_2^2$ then satisfies

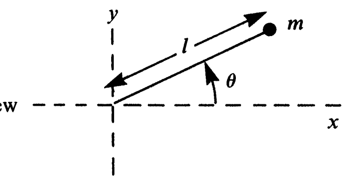
$$\{\tilde H(x,p,w),p^TG(x)\}=0\tag{38}$$

while

$$\{p^TG(x),w_i^*(x,p)\}=0,\qquad i=1,2.\tag{39}$$

Example 2. Consider a mathematical pendulum in space ($\mathbb{R}^3$) with mass $m=1$ and length $l=1$.

Side view


Upside view


Suppose there is a horizontal field by which one can exert a force u_1 in the x-direction and a force u_2 in the y-direction. In spherical coordinates the dynamical equations are

$$(40)\qquad \begin{aligned} \ddot{\theta} &= -u_1 \sin\theta + u_2 \cos\theta, \\ \ddot{\phi} &= -g \sin\phi + u_1 \cos\theta \cos\phi + u_2 \sin\theta \cos\phi \end{aligned}$$

with $(\phi, \theta) \in S^2$ (the unit sphere). Therefore the state space is $M = TS^2$ with local coordinates $x_1 = \phi$, $x_2 = \theta$, $x_3 = \dot{\phi}$, $x_4 = \dot{\theta}$. Once more we take $L(x, u) = \frac{1}{2}(u_1^2 + u_2^2)$. The symmetry vectorfield on M is given in local coordinates by $G(x) = (\partial/\partial x_2)$, with corresponding Hamiltonian $p^T G(x) = p_2$. Then

$$\begin{aligned} &\{p^T f(x, u) - L(x, u), p^T G(x)\} \\ &\quad = \{p_1 x_3 + p_2 x_4 + p_3(-g \sin x_1 + u_1 \cos x_2 \cos x_1 + u_2 \sin x_2 \cos x_1) \\ &\qquad\qquad + p_4(-u_1 \sin x_2 + u_2 \cos x_2) - \tfrac{1}{2}(u_1^2 + u_2^2), p_2\} \\ &\quad = u_1 p_3 \sin x_2 \cos x_1 - u_2 p_3 \cos x_2 \cos x_1 + u_1 p_4 \cos x_2 + u_2 p_4 \sin x_2. \end{aligned}$$

Furthermore

$$(41)\qquad \begin{aligned} y_1 &= \frac{\partial H}{\partial u_1} = p_3 \cos x_2 \cos x_1 - p_4 \sin x_2 - u_1, \\ y_2 &= \frac{\partial H}{\partial u_2} = p_3 \sin x_2 \cos x_1 + p_4 \cos x_2 - u_2. \end{aligned}$$

Hence $\{H(x, p, u), p^T G(x)\} = u_1(y_2 + u_2) - u_2(y_1 + u_1) = u_1 y_2 - u_2 y_1$.

So $(p_2, u_1 y_2 - u_2 y_1)$ is a conservation law satisfying the conditions of Theorems 3 and 4. Hence

$$(42)\qquad \{H^0, p_2\} = 0,$$

$$(43)\qquad \{p_2, u_1^*\} = -u_2^*, \qquad \{p_2, u_2^*\} = u_1^*.$$

Of course, this can be easily checked. Setting $y_1 = y_2 = 0$ in (41), one obtains

$$(44)\qquad u_1^* = p_3 \cos x_2 \cos x_1 - p_4 \sin x_2, \qquad u_2^* = p_3 \sin x_2 \cos x_1 + p_4 \cos x_2$$

in accordance with (43). Furthermore one calculates

$$(45)\qquad H^0(x, p) = H(x, p, u^*(x, p)) = p_1 x_3 + p_2 x_4 - p_3 g \sin x_1 + p_3^2 \cos^2 x_1 + p_4^2$$

and hence (42) is satisfied.

We note that this example, although very close to Example 1, *cannot* be treated by the methods of [4]. This is because there does not exist a smooth feedback $u = \alpha(x, w)$ such that $\{\tilde{H}(x, p, w), p_2\} = 0$; the feedback (37) in rectangular coordinates is *not* defined for $\phi = \theta = 0$. (In Example 1 the origin was excluded from the state space!)

In the above case u_1^* and u_2^* can be immediately computed, thanks to the simple form of $L(x, u)$. However suppose $L(x, u)=\frac{1}{4}(u_1^2+u_2^2)^2$. Then still $\{H(x, p, u), p_2\}= u_1 p_3 \sin x_2 \cos x_1 - u_2 p_3 \cos x_2 \cos x_1 + u_1 p_4 \cos x_2 + u_2 p_4 \sin x_2$ while

$$y_1 = p_3 \cos x_2 \cos x_1 - p_4 \sin x_2 - (u_1^2+u_2^2)u_1,$$

$$y_2 = p_3 \sin x_2 \cos x_1 + p_4 \cos x_2 - (u_1^2+u_2^2)u_2.$$

Hence $\{H(x, p, u), p_2\} = u_1(y_2+u_2(u_1^2+u_2^2)) - u_2(y_1+u_1(u_1^2+u_2^2)) = u_1y_2 - u_2y_1$. Consequently $(p_2, u_1y_2-u_2y_1)$ is still a conservation law. Therefore although u_1^* and u_2^* are not so easy to obtain, one knows a priori that (42) and (43) are satisfied. From (43) one obtains

$$\text{(46)} \qquad \begin{aligned} \frac{\partial^2 u_1^*}{\partial x_2^2} &= \{p_2, \{p_2, u_1^*\}\} = -\{p_2, u_2^*\} = -u_1^*, \\ \frac{\partial^2 u_2^*}{\partial x_2^2} &= \{p_2, \{p_2, u_2^*\}\} = \{p_2, u_1^*\} = -u_2^*. \end{aligned}$$

Consequently as a function of x_2 one knows that u_1^* and u_2^* are of the form $a \sin x_2 + b \cos x_2$, $a, b \in \mathbb{R}$. More generally for any $L(x, u)$ of the form $L(x, u) = h(x_1)\cdot k(\frac{1}{2}(u_1^2+u_2^2))$, with h and k arbitrary smooth functions, one has

$$\begin{aligned} \{p^Tf(x, u) - L(x, u), p_2\} &= u_1\left(y_2 + h(x_1)\frac{dk}{dz}\left(\frac{1}{2}(u_1^2+u_2^2)\right)\cdot 2u_2\right) \\ &\quad - u_2\left(y_1 + h(x_1)\frac{dk}{dz}\left(\frac{1}{2}(u_1^2+u_2^2)\right)\cdot 2u_1\right) \\ &= u_1y_2 - u_2y_1. \end{aligned}$$

So again $(p_2, u_1y_2-u_2y_1)$ is a conservation law.

Example 3. We shall show that in the linear-quadratic case there *cannot* exist quadratic conservation laws (F, F^e) with $F^e=0$, if the system is controllable. Hence the methods of [4] are in this case not applicable. Consider a linear system $\dot{x} = Ax + Bu$ with $L(x, u) = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu + u^TSx$. A linear geometrical symmetry $\dot{x} = Gx$ with G a square matrix corresponds to a quadratic Hamiltonian p^TGx. Calculating,

$$\begin{aligned} &\{p^T(Ax+Bu) - \tfrac{1}{2}x^TQz - \tfrac{1}{2}u^TRu - u^TSx, p^TGx\} \\ &= -p^T(AG-GA)x + p^TGBu + x^TQGx + u^TSGx. \end{aligned}$$

Now suppose $(p^TGx, 0)$ is a conservation law. Then

$$\text{(47)} \qquad AG = GA, \quad GB = SG = 0, \quad QG \text{ skew-symmetric.}$$

The first two equations yield $G(A^kB) = A^kGB = 0$, $k = 0, 1, \cdots$. Hence if (A, B) is controllable necessarily $G=0$! Feedback $u = Fx + Hv$, $\det H \neq 0$, cannot change this situation since $H(x, p, u)$ remains of the same form and (A, B) is controllable if and only if $(A+BF, BH)$ is controllable.

However, there *may* exist linear symmetries with $F^e(y, u) = \frac{1}{2}y^TMy + y^TNu$ (and so $F^e(0, u)=0$). Consider for example the system $\dot{x} = u$ on $\mathbb{R}^n$ with $L(x, u) = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu$, where $Q = Q^T$ and $R = R^T > 0$. The Hamiltonian is $H(x, p, u) = p^Tu - \frac{1}{2}x^TQx - \frac{1}{2}u^TRu$ and the optimal Hamiltonian is obtained by setting $\partial H/\partial u_j = 0$, which yields $u^* = R^{-1}p$, and hence $H^0(x, p) = \frac{1}{2}p^TR^{-1}p - \frac{1}{2}x^TQx$.

Let us look at symmetries for $H^0(x, p)$ of the form p^TFx, with F an $n\times n$-matrix. $\{H^0(x, p), p^TFx\} = p^TR^{-1}F^Tp + x^TQFx$, and hence F has to be such that $R^{-1}F^T$ and QF are skew-symmetric. Now $\{H(x, p, u), p^TFx\} = \{p^Tu - \frac{1}{2}x^TQx - \frac{1}{2}u^TRu, p^TFx\} = u^TF^Tp - x^TQFx = u^TF^Tp$. Let us take $F^e(y, u) = y^TFu$. Since $y = \partial H/\partial u = p - Ru$ we obtain $F^e(y, u) = -u^TRFu + p^TFu$. Because $R^{-1}F^T$, or equivalently, RF has to be skew-symmetric, $F^e(y, u) = p^TFu$, and hence $\{H(x, p, u), p^TFx\} = y^TFu$.

Remark. This last example shows the close connection of our theory with the original Noether theorem on symmetries of Lagrangian functions. This is further investigated in [9].

One of the most pressing questions is now the following. By looking at conservation laws (F, F^e), with $F^e(0, u) = 0$, for the associated Hamiltonian system, can we obtain *all* the symmetries for the optimal Hamiltonian, and if not, which subclass of symmetries do we obtain?

The first part of this question is answered as follows. Let the dimension of the codistribution, generated by taking Poisson brackets of the functions $H(x, p, u)$ for each u, be $k \leqq 2n$ (for simplicity we assume constant dimensions). Then there are exactly $2n-k$ independent functions K_i such that $\{H(x, p, u), K_i\} = 0$. Furthermore we can arbitrarily choose m independent functions F_i^e on $\mathbb{R}^{2n}$ satisfying $F_i^e(0, u) = 0$ for all u. Hence by Theorem 5 there exist *at most* $\min(m, k)$ independent functions F_i on $\mathbb{R}^{2n}$, also independent from the functions K_i, such that there exist functions F_i^e on $\mathbb{R}^{2m}$ in such a way that (F_i, F_i^e) are conservation laws for the Hamiltonian system. Hence, in general we do *not* obtain all the symmetries of the optimal Hamiltonian $H^0(x, p)$.

The second part of the question, which subclass of symmetries do we obtain, is much harder. Let X_F be a symmetry for the optimal Hamiltonian, i.e., $X_F(H^0) = \{F, H^0\} = 0$. Then it follows that

$$\{H(x, p, u), F(x, p)\} = F'(x, p, u) \tag{48}$$

with the function F' satisfying

$$F'(x, p, u^*(x, p)) = 0. \tag{49}$$

Now X_F corresponds to a conservation law for the Hamiltonian system if and only if $F'(x, p, u)$ can be written as a function of $y = \partial H/\partial u$ and u, i.e., if there exists a function $F^e: \mathbb{R}^{2m} \to \mathbb{R}$ such that $F'(x, p, u) = F^e((\partial H/\partial u)(x, p, u), u)$.

PROPOSITION 6. *Let $X_F(H^0) = 0$. Then there exists an $F^e: \mathbb{R}^{2m} \to \mathbb{R}$ such that (F, F^e) is a conservation law for the associated Hamiltonian system if and only if for every $G: \mathbb{R}^{2n} \to \mathbb{R}$ such that $(\partial/\partial u_j)\{H(x, p, u), G(x, p)\} = 0$, $j = 1, \cdots, m$, it follows that $\{\{H(x, p, u), F(x, p)\}, G(x, p)\} = 0$.*

Proof. Let $\partial/\partial u_j\{H, G\} = 0$; then equivalently $X_G(y_j) = X_G(\partial H/\partial u_j) = \{\partial H/\partial u_j, G\} = 0$. Also let $\{H(x, p, u), F(x, p)\} = F'(x, p, u)$. Then $\{\{H(x, p, u), F(x, p)\}, G(x, p)\} = -X_G(F') = 0$, for every such G, implies that $F'(x, p, u)$ only depends on y and u, and hence is of the form $F^e(\partial H/\partial u, u)$. Since $F'(x, p, u^*(x, p)) = 0$, it follows that $F^e(0, u) = 0$. □

By Theorem 4 it is also a necessary condition for a symmetry X_F of H^0 to be obtainable from a conservation law (F, F^e) that F satisfies equations of the form $\{F, u_i^*\} = G_i(u^*)$. This brings us to another interesting point. If we apply *feedback* $u = \alpha(x, v)$, $v \in \mathbb{R}^m$, with the matrix $\partial\alpha/\partial v$ nonsingular, to the system $\dot{x} = f(x, u)$ and the running cost $L(x, u)$, we obtain

$$\dot{x} = \tilde{f}(x, v) := f(x, \alpha(x, v)), \qquad \tilde{L}(x, v) := L(x, \alpha(x, v)) \tag{50}$$

resulting in a new Hamiltonian

$$\tilde{H}(x, p, v) = p^T\tilde{f}(x, v) - \tilde{L}(x, v). \tag{51}$$

Now it is clear that

$$\begin{aligned}\max_v \tilde{H}(x, p, v) &= \max_v p^Tf(x, \alpha(x, v)) - L(x, \alpha(x, v)) \\ &= \max_u p^Tf(x, u) - L(x, u) = \max_u H(x, p, u) = H^0(x, p).\end{aligned} \tag{52}$$

Hence the optimal Hamiltonian $H^0(x, p)$ does not change under feedback, and consequently the symmetries for H^0 remain the same. However the Hamiltonian systems associated respectively to $H(x, p, u)$ and $\tilde{H}(x, p, v)$ are really different. (It is in general not true that by applying feedback $u = \beta(x, p, v)$ to the Hamiltonian system resulting from $H(x, p, u)$ one can obtain the Hamiltonian system corresponding to $\tilde{H}(x, p, v)$.) Consequently the set of conservation laws (F, F^e) for both Hamiltonian systems are in general different. Hence it may happen that for a symmetry X_F of H^0 there exists an F^e such that (F, F^e) is a conservation law for $H(x, p, u)$, while there does *not* exist an $\tilde{F}^e$ such that $(F, \tilde{F}^e)$ is a conservation law for $\tilde{H}(x, p, v)$. Moreover if $u^*(x, p)$ and $v^*(x, p)$ are the optimal controls resulting from maximizing H and $\tilde{H}$, then there may exist functions G_i such that $\{F, u_i^*\} = G_i(u^*)$, but no functions $\tilde{G}_i$ such that $\{F, v_i^*\} = \tilde{G}_i(v^*)$. The following question is therefore worthwhile to investigate.

Question. Let $H(x, p, u) = p^Tf(x, u) - L(x, u)$ be the Hamiltonian of an optimal control problem yielding the optimal Hamiltonian $H^0(x, p) = \max_{u \in \mathbb{R}^m} H(x, p, u)$. Let X_F be a symmetry for H^0, i.e., $X_F(H^0) = 0$. Does there exist a feedback $u = \alpha(x, v)$, $v \in \mathbb{R}^m$, and a smooth function $\tilde{F}^e$ on $\mathbb{R}^{2m}$ such that $(F, \tilde{F}^e)$ is a conservation law for the Hamiltonian system corresponding to $\tilde{H}(x, p, v) = p^T\tilde{f}(x, v) - \tilde{L}(x, v)$?

If the above question can be answered affirmatively, then in a sense all the symmetries for the optimal Hamiltonian can be recovered from symmetries of an associated Hamiltonian system.

Remark. The above question is also related to the problem of bringing $H(x, p, u)$ into some kind of normal form by feedback transformations $u = \alpha(x, v)$ and state space transformations. If we allow for the *larger* class of transformations $u = \alpha(x, p, v)$ and take the usual assumption that $(\partial^2 H/\partial u_i\, \partial u_j)$ is nonsingular (say for simplicity negative definite), then the Morse Lemma yields for $H(x, p, u)$ the normal form $\bar{H}(x, p) - \sum_{j=1}^m u_j^2$. Hence we end up with the Hamiltonian system

$$\begin{aligned}\dot{x}_i &= \frac{\partial \bar{H}}{\partial p_i}(x, p), \\ \dot{p}_i &= -\frac{\partial \bar{H}}{\partial x_i}(x, p),\end{aligned} \qquad y_j = u_j. \tag{53}$$

In this degenerate case $H^0(x, p) = \bar{H}(x, p)$, and $\{\bar{H}(x, p) - \sum_{j=1}^m u_j^2, F(x, p)\}$ equals a function $F^e((\partial H/\partial u), u)$ if and only if $\{\bar{H}, F\} = 0$. It would be interesting to extend the Morse Lemma to the smaller class of transformations $u = \alpha(x, v)$.

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